Convolution:

Let $f \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ $g \in L^p$ with Lebesgue measure. Then

$$f * g(x) = \int f(x-y) g(y)\, dy \quad \text{is well defined} \quad \text{a.s. } \lambda$$

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p \qquad \text{where } p \in [1, \infty].$$

Pf: Assume $1 \leq p < \infty$. Assume $f \neq 0$ a.s.

$$\left(\int |g(x-y) f(y)|\, dy \right)^p = \left(\int |g(x-y)| \frac{|f(y)|}{\|f\|_1}\, dy \right)^p \|f\|_1^p \underset{\text{Jensen}}{\leq} \|f\|_1^p \int |g(x-y)|^p \frac{|f|}{\|f\|_1}\, dy$$

$$= \|f\|_1^{p-1} \int |g(x-y)|^p |f(y)|\, dy$$

Then $$\|f * g\|_p^p = \int \left(\int |g(x-y) f(y)|\, dy \right)^p dx \leq \|f\|_1^{p-1} \int\int |g(x-y)|^p |f(y)|\, dy\, dx$$

$$= \|f\|_1^{p-1} \|g\|_p^p \|f\|_1 = \left(\|f\|_1 \|g\|_p \right)^p.$$

Prop: Let $p \in [1, \infty)$ $q = p^* \in (1, \infty]$. Then $|f * g(x)| \leq \|f\|_p \|g\|_q$ $\forall x \in \mathbb{R}^d$ and $f * g$ is uniformly continuous on $\mathbb{R}^d$.

HW. Pf: This is a bit of work, but not too much.

σ-field generated by an rv

Let $X : \Omega \to (E, \Sigma)$ a meas. rv on $(\Omega, \mathcal{F})$.

$$\sigma(X) := \{ X^{-1}(B) : B \in \Sigma \} \subset \mathcal{F}$$

↑ this is just a σ-algebra

In general for a collection $(X_i)_{i \in I}$ $\quad \sigma((X_i)_{i \in I}) = \sigma(\{ X_i^{-1}(B_i) : B_i \in \Sigma_i \})$

smallest σ algebra containing this union of σ algebras.

Prop Let $X : \Omega \to E$ be meas and let $Y : \Omega \to \mathbb{R}$ be meas. real valued. Then $\exists$ $f : E \to \mathbb{R}$ measurable so that $Y = f(X) \iff Y$ is $\sigma(X)$ meas.

Pf: If $\exists f$ st $Y = f(X)$ then $Y^{-1}(A) = \{ \omega : f(X(\omega)) \in A \}$

$$= \{ \omega : X(\omega) \in f^{-1}(A) \} = X^{-1}(f^{-1}(A))$$

since $f^{-1}(A) \in \Sigma$, $X^{-1}(f^{-1}(A)) \in \sigma(X)$ by definition.

So the only hard case is if Y is $\sigma(X)$ meas. If Y is simple then

$$Y(\omega) = \sum a_i 1_{E_i} \text{ where } E_i \in \sigma(X) \Rightarrow E_i = X^{-1}(F_i) \text{ where } F_i \in \Sigma$$

$$= \sum a_i 1_{X^{-1}(F_i)}(\omega) = \sum a_i 1_{F_i}(X(\omega))$$

So set $f(t) = \sum a_i 1_{F_i}(t)$

In general, we use simple approximation: if Y is real valued $\exists$ Y_n simple st $Y_n \to Y$ and we can write $Y_n = f_n(X)$. Define

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if it exists.} \\ 0 & \text{otherwise} \end{cases}$$

But we know $\lim f_n(X(\omega)) = \lim Y_n(\omega) = Y(\omega)$ exists for all ω. Therefore $Y(\omega) = f(X(\omega))$

<u>Covariance</u>: $\quad Cov(X,Y) = E[(X-EX)(Y-EY)]$

$$= E[XY] - E[X]E[Y]$$

If $Z \in \mathbb{R}^d$ is a random vector then $(K_Z)_{ij} = Cov(Z_i, Z_j)$ is a matrix that is symmetric.

Moreover $\quad \sum_{i,j} u_i (K_Z)_{ij} u_j = \sum_{ij} E[u_i u_j (Z_j - E_j)(Z_i - EZ_i)]$

$$= E\left[\left(\sum_{i=1}^{N} u_i (Z_i - E_i Z_i)\right)^2\right] \geq 0$$

Characteristic functions

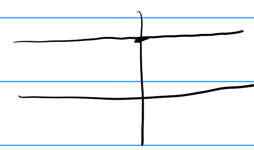
$E[e^{itX}] = \Phi(t)$ determines the law of X.

Prop If $X \sim N(\mu, \sigma^2)$ then $\phi(t) = E[e^{itX}]$

Pf: $$\int e^{itx} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dx$$

$$x^2 - 2\mu x - 2it\sigma^2 x + \mu^2 = (x-(\mu+it\sigma^2))^2 + \mu^2 - (\mu+it\sigma^2)^2$$
$$= (x-(\mu+it\sigma^2))^2 + t^2\sigma^4 - 2\mu it\sigma^2$$

$$= e^{i\mu t - \frac{t^2\sigma^2}{2}} \int \frac{e^{-\frac{(x-(\mu+it\sigma^2))^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dx$$

That is the integral over  in the complex plane.

Let $g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

If X_σ has $N(0,\sigma^2)$ distribution, we showed $E[f(X_\sigma)] \to f(0)\ \forall f \in C_b(\mathbb{R})$

Let μ be the law of X

Define $$f_\sigma(x) = \int g_\sigma(x-y)\,\mu(dy)$$

$$\mu_\sigma(dx) = f_\sigma(x)\,\mu(dx)$$

We know μ is characterized by its values on intervals and one can show this is equivalent to knowing $\int \varphi\,d\mu\ \forall\ \varphi \in C_b(\mathbb{R})$. Let

$$\Phi_X(t) = \int e^{itx}\,\mu(dx)$$ (also known as the Fourier transform)

$$\sqrt{2\pi\sigma^2}\, g_\sigma(x) = e^{-x^2/2\sigma^2} = \int e^{itx}\, g_{1/\sigma}(\xi)\,d\xi$$

"The Gaussian is invariant under Fourier transformation" and hence μ_6

Will show: i) f_6, can be written entirely in terms of $\Phi_X(t)$

2) $\int \varphi \mu_6(dx) \to \int \varphi(x) \mu(dx)$ as $\sigma \downarrow 0$

So the procedure is, given Φ determine μ_6 and then use that to determine $\mu(a,b)$.

$$f_6(x) = \int g_6(x-y)\, \mu(dy) = \frac{1}{\sqrt{2\pi\sigma^2}} \int\int e^{i\xi(x-y)} g_{1/6}(\xi)\, d\xi\, \mu(dy)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{i\xi x} g_{1/6}(\xi)\, \Phi_X(-\xi)\, d\xi \quad \Rightarrow \quad f_6 \text{ can be determd by } \Phi_X$$

Here we used Fubini since μ is a prob measure, $|e^{i\xi x}| \leq 1$ and $g(\xi)$ is integrable wrt 6. So can use Tonelli to show integrability.

symmetry in x-y

$$\int \varphi \mu_6(dx) = \int \varphi(x) \Big(\underbrace{\int g_6(x-y)\, \mu(dy)}_{f_6(x)} \Big) dx = \int \int \varphi(x) g_6(y-x)\, dx\, \mu(dy)$$

$$= \int g_6 * \varphi\, \mu(dy) \qquad g_6 * \varphi(y) = E[\varphi(y - X_6)]$$

$$\to \varphi(y) \quad \text{as } \sigma \downarrow 0$$

Moreover since φ is bounded we can use doM to show

$$\int \varphi \mu_6(dx) \to \int \varphi(y) \mu(dx)$$